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QUALITATIVE INVESTIGATION OF A SYSTEM OF THREE DIFFERENTIAL EQUATIONS IN THE THEORY OF PHASE SYNCHRONIZATION

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A nonlinear system of three differential equations with three parameters defining the dynamics of a search system for phase synchronization is considered. Qualitative analysis of the system is carried out with the use of Liapunov functions, systems of matching, surfaces without contact, and local theory of bifurcation of multidimensional dynamic systems. It is established that the effective parameter range is determined by the bifurcation of the saddle separatrix loop.

1. Introduction. We consider a system of three differential equations of the form

$$\varphi^{\cdot} = y, \quad y^{\cdot} = v - \frac{a}{b}[F(\varphi) - \gamma], \quad v^{\cdot} = -\frac{1}{b}v + \frac{1}{b}\left(\frac{a}{b} - 1\right)[F(\varphi) - \gamma] \tag{1.1}$$

where a, b and γ are parameters of function $F(\varphi) \in c^k$ $(k \geqslant 2)$. The system satisfies the following conditions:

$$F(\varphi) = F(\varphi + 2\pi), \quad -F(\varphi) = F(-\varphi), \quad F_\varphi'(\varphi) > 0, \quad \varphi \in (-\varphi_0, \varphi_0), \; F_\varphi'(\varphi) < 0, \quad \varphi \in (\varphi_0, 2\pi - \varphi_0), \; F_\varphi'(\varphi_0) = 0, \; F(\varphi_0) = 1 \tag{1.2}$$

System (1.1) is specified in the cylindrical phase space $G = S^1 \times R^2 (\varphi \in S^1; y, v \in R^2)$. Cylindricity of the phase space implies the possibility of the existence in it of recurrent motions not only of the oscillatory type (motions that remain recurrent in the R^3 ($\varphi \in R$; y, $v \in R^2$) covering space for G),but also of the rotational type (motions that loose the recurrency property in the covering space) [1, 2]. We refer to recurrent trajectories of the oscillatory type, which differ from the equilibrium state, as o-trajectories, and to those of the rotational type in region $y > 0$ ($y < 0$) as the φ^1 (φ^2) trajectories.

System (1.1) is a mathematical model of an astatic system of phase synchronization in conditions of linear variation of the input signal frequency [3] or of continuous search with respect to frequency [4 – 6]. In (1.1) γ is the relative rate of frequency change, and A and b are nonnegative parameters of the control system.

The capture mode in a phase synchronization system is simulated by the motion of the representing point along the trajectory of system (1.1) toward the equilibrium state O_1 ($\varphi = \text{const}$, $y = v = 0$). Investigation of dynamics of the search system of phase synchronization consists in the determination of the critical rate of frequncy variation $\gamma = \gamma_1(a, b)$ which defines the interval $|\gamma| < \gamma_1(a, b)$ within which a capture mode is realized for any initial conditions in region $y < 0$. This requires a qualitative investigation of system (1.1).

In the degenerate case of $b = 0$ (and also in the case of a small parameter [7 – 9] at the derivative for $b = \mu \ll 1$) the qualitative analysis of system (1.1) with variables φ, y and $z = v - (a / b)[F(\varphi) - \gamma]$ for $F(\varphi) = \sin\varphi$ is presented in [5], and is extended in [6], Theorem 3, to the case of (1.2). The critical rate $\gamma_1(a, b)$ and the time of transient processes was determined numerically in [10] for $F(\varphi) = \sin\varphi$ and certain values of a and b.

The qualitative analysis of system (1.1) is effected here for any $F(\varphi)$ satisfying (1.2) and any $b > 0$.

2. Preliminary investigation. Let us consider system (1.1) in the parameter region D where $a > 0$, $b > 0$ and $\gamma \geqslant 0$. Parameter γ is assumed to be nonnegative, owing to the symmetry of system (1.1), hence the substitution $\gamma = -\gamma$, $\varphi = -\varphi^\circ$, $y = -y^\circ$ and $v = -v^\circ$ does not alter system (1.1).

Let us consider the control function $w_1 = v^2 / 2$ outside the phase space of system (1.1)

$$G_v = \{\varphi \in S^1; \; y \in R; \; -|ab^{-1} - 1|(1 + \gamma \operatorname{sgn}(a - b)) < v < |ab^{-1} - 1|(1 - \gamma \operatorname{sgn}(a - b)), \; \gamma \leqslant 1; \; -(ab^{-1} - 1)(1 + \gamma) < v < 0, \; \gamma > 1, \; b < a; \; 0 < v < (1 - ab^{-1})(1 + \gamma), \; \gamma > 1, \; b > a\}$$

In virtue of system (1.1), the derivative of w_1 is negative, hence we have the following statement.

Statement 1. Region G_v is stable, since for $t \to +\infty$ all trajectories of system

(1.1) penetrate from region $G \setminus G_v$ into it.

For $b = a$ region G_v degenerates into the cylinder $v = 0$, the only one asymptotically stable integral surface of system (1.1). Motions on the integral cylinder $v = 0$ are determined by the conservative system whose integral is

$$H(\varphi, y) = \frac{y^2}{2} - \frac{a}{b}\int_0^{\varphi} [\gamma - F(\xi)]\, d\xi = h \qquad (2.1)$$

Let us consider in G_v the set of cylinders $w_2 = y + bv = \text{const}$. Since in virtue of system (1.1) and in accordance with (1.2), the derivative of function w_2 is positive for $\gamma > 1$, when $w_2 = -[F(\varphi) - \gamma] > 0$, and the cylinders $w_2 = \text{const}$ are surfaces without contact. Then, using Statement 1, we obtain the following statement.

Statement 2. For $\gamma > 1$ system (1.1) has no singular trajectories, for $t \to +\infty$ all of its trajectories penetrating (or found in) G_v tend to infinity $y \to +\infty$.

3. Equilibrium states. The equilibrium states of system (1.1) lie in G_v along the circumference $y = v = 0$ and are determined by two roots: $\varphi_1(\gamma) \in [0, \varphi_0)$ and $\varphi_2(\gamma) \in (\varphi_0, \pi]$ of equation $\gamma - F(\varphi) = 0$. For $0 \leqslant \gamma < 1$ system (1.1) has two states of equilibrium: O_1 ($\varphi = \varphi_1(\gamma)$, $y = v = 0$) and O_2 ($\varphi = \varphi_2(\gamma)$, $y = v = 0$); for $\gamma = 1$ it has one equilibrium state O_0 ($\varphi = \varphi_0$, $y = v = 0$) which vanishes for $\gamma > 1$. The characteristic equation for equilibrium states is of the form

$$\varkappa^3 + b^{-1}\varkappa^2 + ab^{-1}F_\varphi'(\varphi_i)\varkappa + b^{-1}F_\varphi'(\varphi_i) = 0, \quad i = 0, 1, 2 \qquad (3.1)$$

In virtue of (1.2) $F_\varphi'(\varphi_1) \equiv m(\gamma) > 0$, $F_\varphi'(\varphi_2) \equiv -n(\gamma) < 0$ and $F_\varphi'(\varphi_0) = 0$. According to (3.1) for $b < a$, O_1 is either a stable node ($\varkappa_i < 0$, $i = 1, 2, 3$), or a stable focal point ($\varkappa_1 < 0$, $\operatorname{Re}\varkappa_{2,3} < 0$), and for $b > a$ it is either a saddle ($\varkappa_1 < 0$, $\varkappa_{2,3} > 0$) or a saddle-focus ($\varkappa_1 < 0$, $\operatorname{Re}\varkappa_{2,3} > 0$). In the parameter region $b > f(a, n(\gamma)) \geqslant a$, where $f(a, n(\gamma))$ is the positive root of equation $27n(\gamma)b^2 - an(\gamma)[4a^2n(\gamma) + 18]b - a^2n(\gamma) - 4 = 0$ in b, the equilibrium state O_2 is a saddle-focus ($\varkappa_1 > 0$, $\operatorname{Re}\varkappa_{2,3} < 0$), and in region $b < f(a, n(\gamma))$ it is a saddle ($\varkappa_1 > 0$, $\varkappa_{2,3} < 0$).

According to [11] the equilibrium state O_2 is traversed by two manifolds: a one-dimensional curve consisting of O_2 and two unstable separatrices S_1^- and S_2^- outgoing into regions $G_v^1 = G_v^2\ (y > 0)$ and $G_v^2 = G_v\ (y < 0)$, respectively, and the two-dimensional stable surface S^+.

For $\gamma = 1$, O_0 is a complex equilibrium state with two zero roots. Since for $\gamma = 1$ surfaces $w_2 = \text{const}$ are in contact with trajectories of system (1.1) only at the cylinder which passes through O_0, we obtain from Statement 2 that for $\gamma = 1$ all trajectories of system (1.1), except those in contact with O_0, tend to infinity when $t \to +\infty$. Below we assume that $0 \leqslant \gamma < 1$.

Let us introduce definitions of the structure of phase space separation into trajectories of system (1.1).

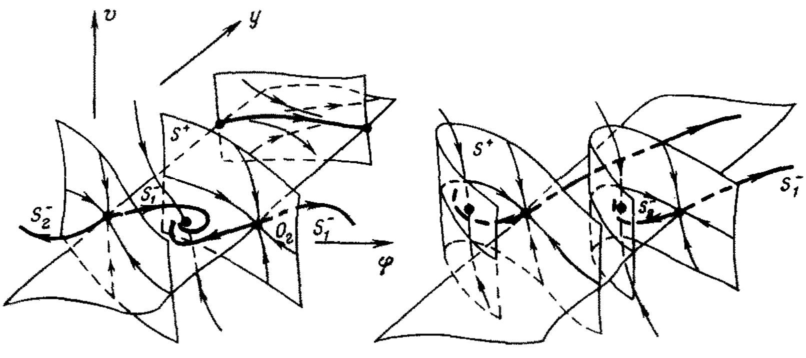
Structure K corresponds to asymptotic stability O_1 throughout the space G_v except the saddle O_2 and its stable separatrix surface S^+. The manifolds $S_{1,2}^-$ and S^+ are necessarily disposed so that $S_{1,2}^-$ tends to O_1, and S^+ tends to infinity.

Structure K^- does not contain o- and φ-trajectories, O_1 is unstable, $S_{1,2}^-$ tend to infinity, and the trajectories on surface S^+ tend for $t \to -\infty$ to O_1 and escape to infinity.

Structure L is structure K complemented by the saddle φ^1-cycle (Fig. 1, a).

Structure L^- is structure K^- complemented by the stable φ^2-cycle.

Structure M does not contain o- and φ-trajectories, S_1^- tends in region G_v^1 to infinity, S_2^- tends to O_1 and $S^+ \cap G_v$ tends in region G_v^2 to infinity encompassing the stable equilibrium state O_1 and restricting its region of pull (Fig. 1, b).



a b

Fig. 1

Structure M^- does not contain o- and φ-trajectories, S_1^- and S_2^- move to regions G_v^1 and to infinity, a part of trajectories on $S^+ \cap G_v$ tend for $t \to -\infty$ to O_1, while another part moves to regions G_v^2 and infinity.

4. Liapunov functions. Let us consider functions of the form

$$V_i = \frac{(-1)^i}{2} \left\{(b-a)(y+bv)^2 - b^3v^2 + 2(b-a)\int_{\varphi_1}^{\varphi} [F(\xi) - \gamma]\, d\xi\right\} \qquad (4.1)$$

$$(-1)^i (b-a) > 0 \qquad (i = 1, 2)$$

whose derivatives satisfy the inequalities

$$(-1)^i V_i^{\cdot} = b^2v^2 \geqslant 0 \quad (i = 1,\ 2) \qquad (4.2)$$

in virtue of system (1.1).

For $\gamma = 0$, V_i are periodic Liapunov functions [1, 2], hence inequalities (4.2) prove the following statement.

Lemma 1. (1) For $\gamma = 0$ and $b < a$ system (1.1) has structure K; (2) for $\gamma = 0$ and $b > a$ it has structure K^-.

For $\gamma > 0$ functions (4.1) are not periodic with respect to φ. We assume that they are determinate in the covering space R^3, not in G_v. We introduce the notation $c_i = V_i(\varphi_2, 0, 0)$.

Lemma 2. (1) System (1.1) has no o-trajectories; (2) for $\gamma > 0$ and $b < a$ region $\Omega^+ :\ V_1(\varphi, y, v) < c_1,\ \varphi < \varphi_2$ belongs to the attraction region of stable equilibrium state O_1; and (3) for $\gamma > 0$ and $b < a$ ($b > a$) system (1.1) has no φ^2-(φ^1)-trajectories, and for $\gamma > 0$ infinity in space G_v^1 is stable and in G_v^2 unstable.

Proof. (1) In virtue of (4.2) and statement 2.1 the trajectories of system (1.1) cannot reach one and the same level $V_i = \text{const}$, hence system (1.1) has no o-trajectories.

(2) The statement follows from that, that in region Ω^+ function V_1 is positively determinate and satisfies (4.2). (3) For $\gamma = 0$ surfaces $V_i = c(c > c_i)$ are 2π-periodic with respect to φ, and the derivatives $V_i^{\cdot}$ satisfy in virtue of system (1.1) in region G_v for $\gamma \neq 0$ the inequalities

$$(-1)^i V_i^{\cdot} = b^2v^2 + \gamma (b-a) y > 0, \quad (-1)^i y > 0$$

$$(-1)^i V_i^{\cdot} < 0, \quad (-1)^i [y + (b-a)(1+\gamma)^2 \gamma^{-1}] < 0 \quad (i = 1, 2)$$

which proves the last statement of the lemma.

5. Systems of matching. Let us consider the auxilliary system A_i of the form (2.1), where $\gamma_i \equiv \gamma - | b - a | a^{-1} [1 - (-1)^i\gamma]$ is substituted for parameter γ and in the parameter space $(-1)^i (b-a) > 0$ $(i = 1, 2)$. For $\gamma < b^{-1} [a + (-1)^i (b-a)]$ systems A_i have each two equilibrium states, one with respect to the center ($\varphi = \varphi_{1i}$, $y = 0$) and the second with respect to the saddle ($\varphi = \varphi_{2i}$, $y = 0$) $(i = 1, 2)$. We introduce surfaces W_i determinate in G_v and formed by pieces of systems A_i separatrices of the form

$$W_i = \left\{H(\varphi, y, \gamma_i) = -ab^{-1} \int_0^{\varphi_{2i}} [\gamma_i - F(\xi)]\, d\xi, \ (-1)^i y \leqslant 0, \ v \in G_v\right\}$$

$(i = 1, 2)$

Matching the vector fields of system (1.1) and A_i with the use of Statement 1, we find that the trajectories of system (1.1) intersect surfaces W_i in the direction of increasing y without contact with the latter.

Lemma 3. (1) In the parameter region $\Delta_1 : ab^{-1} - 1 < \gamma < 1$, $a2^{-1} < b < a$ system (1.1) has structure M, and (2) in the parameter region $\Delta_2 : 1 - ab^{-1} < \gamma < 1$, $b > a$ a system (1.1) has structure M^-.

Proof. The disposition of surface W_1 (W_2), which is passing across straight line $\varphi = \varphi_{21}$ (φ_{22}), $y = 0$, tends in region G_v^1 (G_v^2) to infinity, implies the absence of φ^1- (φ_2-) trajectories. Considering that in the parameter region Δ_1 (Δ_2) the saddle O_2 lies in virtue of inequality $\varphi_2 < \varphi_{2i}$ under (above) the surface W_1 (W_2), we find that S_1^- (S^+), which in region G_v^1 (G_v^2) intersects W_1 (W_2), tends to infinity, and S^+ (S_2^-) remains under (above) surface W_1 (W_2), while moving into region G_v^2 (G_v^1). Then, using Lemma 2, we establish that in the parameter region Δ_1 (Δ_2) system (1.1) has the structure M (M^-).

6. Bifurcations and φ-cycles. We denote by Γ_1 (Γ_2) the saddle (saddle-focus) separatrix loop O_2, produced by separatrix S_1^- (S_2^-) emanating from O_2 and returning to it by executing in region G_v^1 (G_v^2) a turn with respect to φ.

Theorem 1. There exist functions $\gamma = \gamma_1(a, b)$ and $\gamma = \gamma_2(a, b)$ associated with the existence of loops Γ_1 and Γ_2, respectively, which satisfy conditions

$$0 < \gamma_1(a, b) < \begin{cases} 1, & 0 < b < a2^{-1} \\ ab^{-1} - 1, & a2^{-1} \leqslant b < a \end{cases}; \quad \gamma_1(a, a) = 0 \tag{6.1}$$

$$0 < \gamma_2(a, b) < 1 - ab^{-1}, \quad b > a, \quad \gamma_2(a, a) = 0 \tag{6.2}$$

Proof. Lemmas 1 and 2 imply that for $\gamma \geqslant 0$ manifolds $S_{1,2}^-$ and S^+ intersect the plane $\varphi = \varphi_1$. Let y_1^- and v_1^- (y_2^- and v_2^-) be the coordinates of the point at which separatrix S_1^- (S_2^-) intersects for the first time the plane $\varphi = \varphi_1$ from the side $\varphi <$

φ_1 $(\varphi > \varphi_1)$, and let $y_1^+(v)$ $(y_2^+(v))$ be equations of the curve along which surface S^+ intersects for the first time the plane $\varphi = \varphi_1$ from the side $\varphi > \varphi_1$ $(\varphi < \varphi_1)$. We introduce distances $\rho_1(\gamma, a, b) = y_1^- - y_1^+(v_1^-)$ and $\rho_2(\gamma, a, b) = y_2^- - y_2^+(v_2^-)$ which are continuous functions of parameters determinate in region $1 > \gamma \geqslant 0$. In virtue of Sects. 2, 3 and of lemmas, functions $\rho_i(\gamma, a, b)$ $(i = 1, 2)$ satisfy conditions

$$\begin{gathered}\rho_i(0, a, b) < 0\ (-1)^i (b-a) > 0,\ \rho_i(0, a, a) = 0 \\ \rho_i(\gamma, a, b) > 0\ (\gamma \in \Delta_i \text{ и } (-1)^i (b-a) > 0),\ \lim_{\gamma\to 1} \rho_i(\gamma, a, b) > 0\end{gathered} \tag{6.3}$$

Taking into account (6.3) and the continuity of $\rho_i(\gamma, a, b)$ and using Cauchy's theorem about zeros of functions, we establish the existence of functions $\gamma_i(a, b)$ as the solutions of equations $\rho_i(\gamma, a, b) = 0$ $(i = 1, 2)$, which satisfy conditions (6.1) and (6.2), respectively. The theorem is proved.

Remarks. 1) Theorem 1 does not prove the single-valuedness of functions γ_i, which is assumed below for simplicity of exposition.

2) According to [5, 6] there exists the limit relationship $\lim_{b\to 0} \gamma_1(a, b) = \gamma^\circ(a)$, where $\gamma^\circ(a)$ is the bifurcation curve of the separatrix loop of the degenerated two-dimensional system [6].

Theorem 2. For $b < a$ the saddle φ^1-cycle is generated at infinity with parameter γ increasing from zero. For $\gamma = \gamma_1(a, b)$ this cycle descends in region G_v^1 and merges with loop Γ_1.

Proof. We introduce region g_1 periodic with respect to φ and homeomorphic to a torus, whose boundary consists of the following surfaces:

$$u_1^+ : v = b^{-1}(a-b)(1-\gamma), \quad u_2^+\ v = -b^{-1}(a-b)(1+\gamma), \quad u_1^- : V_1(\varphi, y, v) = c,\ \gamma = 0$$

and u_2^-, where u_2^- is a surface without contact, periodic with respect to φ, lying in region G_v^1 below u_1^-, intersected by trajectories of system (1.1) in the direction of decreasing y, and existing for $\gamma < \gamma_1(a, b)$ and $b < a$ because of the disposition of manifolds S_1^- and S^+ $(\rho_1(\gamma, a, b) < 0)$. Let T be the image of the part of plane p: $\{\varphi = \varphi^\circ = \text{const}, y > y^\circ$ and $v \in (u_1^+, u_2^+)\}$ onto itself, generated by trajectories of system (1.1). Using Statement 1 and Lemma 2, we find that the vector field of the image of T at the boundary l^s of the simply-connected region $g^s = p \cap g_1$ is oriented as follows: along curves $u_i^- \cap p$ $(i = 1, 2)$ it is directed into the exterior of region g^s and along curves $u_i^+ \cap p$ $(i = 1, 2)$ into region p (either g^s or $p \setminus g^s$). As the result we have ind $(T)_{l^s} = -1$, hence the image of T in region g^s has at least one stationary saddle point related to the saddle φ^1-cycle of system (1.1). Since by Lemmas 1 and 2 u_1^- for $\gamma \to 0$ departs to infinity, and for $\gamma = 0$ system (1.1) has the K-structure, hence for $\gamma \to 0$ the saddle φ^1-cycle tends to infinity. Owing to the relationships between roots of Eq. (3.1) the conditions of the theorem in [12], which states that for $\gamma = \gamma_1(a, b)$ the saddle φ^1-cycle merges with loop Γ_1, are satisfied for O_2 when $b < a$.

Corollary. Lemma (4.2) and Theorems 1 and 2 imply that system (1.1) has in the parameter region $d_1 : 0 < \gamma < \gamma_1(a, b)$ and $b < a$ (Fig. 2) structure L and in region $d_2 : \gamma_1(a, b) < \gamma < 1$ and $b < a$ its structure is M.

Theorem 3. (1) For $b > a$ and parameter γ increasing from zero a stable φ^2-cycle is generated from infinity, which in region $\gamma < \gamma_2(a, b)$ moves upward over region G_v^2. (2) In region δ_1: $\{a^{-2} > \sup_{\gamma\in[0,1)} n(\gamma) \equiv n^\circ$ and $b < b_1\}$, where

b_1 is the positive root of the equation $b = -2^{-1}a + [4^{-1}a^2 + 2(n(\gamma_2(a, b)))^{-1}]^{1/2}$, the stable φ^2-cycle merges for $\gamma = \gamma_2(a, b)$ with loop Γ_2. In region δ_2: $\{a^{-2} < n^0$ and $b < b_2\}$, where b_2 is the positive root of equation $b = f(a, n(\gamma_2(a, b)))$, there exists region $d_k \subset \delta_2$ for whose points system (1.1) has two φ^2-cycles: one stable and one saddle. In regions $\delta_3 : \{a^{-2} > n^\circ$ and $b > b_1\}$, $\delta_4 : \{a^{-2} = n^\circ$ and $b > a\}$. and $\delta_5 : \{a^{-2} < n^\circ$ and $b > b_2\}$ there exists a parameter space d_c, for whose points system (1.1) has a denumerable number of saddle φ^2-cycles.

Proof. (1) Statement 1, Lemma 2, and the disposition of manifolds S_1^- and S^+ in G_v^2 imply that for $\gamma < \gamma_2(a, b)$ $(\rho_2(\gamma, a, b) < 0)$ there exists region $g_2 \subset G_v^2$ periodic with respect to φ, homeomorphic to a torus which is mapped onto itself by trajectories of system (1.1). Using Brauer's theorem, we conclude from this, that at least one stable φ^2-cycle must exist in g_2. Since for $\gamma = 0$ system (1.1) by Lemma 1 has structure K^-, the φ^2-cycle tends to infinity when $\gamma \to 0$. (2) Using [12, 13] and analyzing the relationship between roots of Eq. (3.1) for O_2, we establish that in region δ_1 (δ_2) a stable (saddle) φ^2-cycle is generated from loop Γ_2 with γ decreasing (increasing) at transition through γ_2, while in regions δ_i $(i = 3, 4, 5)$ a denumerable number of saddle φ^2-cycles exists in the neighborhood of Γ_2. By Lemma 3 system (1.1) has no φ-cycles in region Δ_2, hence all generated φ^2-cycles must vanish with increasing γ. From this we find the φ^2-cycle generated from infinity merges in region δ_1 with loop Γ_2; that in region δ_2 curve $\gamma = \gamma_k$ associated with the bifurcation of multiple φ^2-cycles which together with curve $\gamma = \gamma_2$ constitute the region of two φ^2-cycles d_k: $\{\gamma_2 < \gamma < \gamma_k\}$; and that bifurcation curves $\gamma = \gamma_c^j$ $(j = 1, 2)$, which are the boundaries of the denumerable set of φ^2-cycles $d_c : \{\gamma_c^1 < \gamma < \gamma_c^2\}$, exist in regions δ_i $(i = 3, 4, 5)$. The theorem is proved.

Corollary. The region of $b > a$ is subdivided for $a^{-2} \geqslant n^\circ$ by the bifurcation curves γ_2, γ_k, γ_c^j $(j = 1, 2)$ into regions d_c, $d_3 : \{0 < \gamma < \gamma_2\} \setminus d_c$ and d_4 : $\{\gamma_2 < \gamma < 1\} \setminus \{d_c \cup d_k\}$, and for $d^{-2} < n^\circ$ into regions d_3, d_4, d_c and d_k (Fig. 2). By Lemmas 2 qnd 3 and Theorems 1 and 3 system (1.1) has structure L^- in region d_3, structure M^- in region d_4, structure M^- complemented by two φ^2-cycles in region d_k, and in region d_c it has a denumerable set of φ^2-cycles with manifolds $S_{1,2}^-$ and S^+ disposed as in structure L^- (M^-) for $\gamma < \gamma_2$ ($\gamma > \gamma_2$).

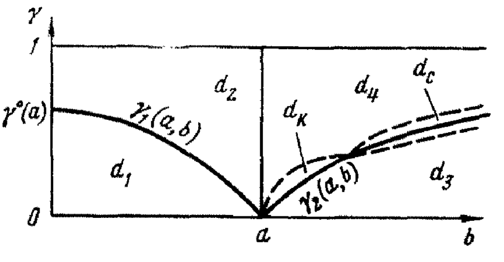


Fig. 2

Remark. Parameter regions δ_i $(i = 1, 2, \ldots, 5)$ obviously exist in Theorem 3, if $n(\gamma)$ is a monotonic function with one maximum at $\gamma = 0$ (e.g. $n(\gamma)$ in the case of $F = \sin\varphi$). In the case of nonmonotonic $n(\gamma)$ some of the δ_i and, consequently, d_k and d_c may be absent.

We note in conclusion that the working parameter region of the system of phase synchronization is the d_1 region which corresponds to structure L. The critical rate of frequency change — the boundary of region d_1 — $\gamma = \gamma_1(a, b)$ — is determined by the bifurcation of loop Γ_1.

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